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COMMENT

On the soliton generation in optical fibres

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Abstract. This is a comment on a recent paper by Burzlaff. His paper is devoted to the investigation of optical soliton generation for special input pulses within the framework of the non-linear Schrödinger equation. This comment demonstrates that the main result by Burzlaff, namely the threshold condition for soliton generation, may be obtained in a general form for any initial pulse with a constant phase (and its Galilei transformation). In particular, our general result includes the Burzlaff results. We note also that the formula obtained is applicable to the sine-Gordon equation with a special type of initial condition.

In a recent paper by Burzlaff (1988a) the generation of optical solitons from two families of initial envelope functions was discussed. He found that, for a purely imaginary initial envelope function of width a and height b, the soliton number of soliton bound states is an integer smaller than $\frac{1}{2}+ab/\pi$ (this result was obtained earlier by Manakov (1973)), and for the initial envelope function $\mathrm{i}\beta\exp(-\alpha|x|)$ that is equal to the number of intersections of the Bessel functions $J_{-1/2}$ and $\pm J_{1/2}$ below β/α, which is an integer smaller than $\frac{1}{2}+2\beta/\alpha\pi$. The main result by Burzlaff for the above-mentioned input pulses may be rewritten as follows: the soliton number N is

$$N=\langle\tfrac{1}{2}+F/\pi\rangle \tag{1}$$

where

$$F=\int_{-\infty}^{\infty}|u(x,0)|\,\mathrm{d}x \tag{2}$$

and $\langle\ldots\rangle$ denotes an integer smaller than the argument. In his recent report Burzlaff (1988b) also considered the super-Gaussian initial pulse, which describes the real laser pulse:

$$u(x,0)=A_0\exp[-\tfrac{1}{2}(1-\mathrm{i}\alpha)(x/\sigma)^{2m}] \tag{3}$$

and obtained some analytical results for it.

Our comment aims to obtain the threshold condition of soliton generation (1) and (2) for a rather wide class of initial pulses including (3) at $\alpha=0$. Our result connects the envelope of the input pulse with the number of generated solitons that should help to choose or build the laser best suited to injecting solitons into optical fibres at lowest cost.

It is well known that the normalised envelope of an electric field in a monomode optical fibre, denoted as u, satisfies the non-linear Schrödinger (NS) equation

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2 u=0 \tag{4}$$

where x and t are the normalised coordinates along the fibre and the time in the group-velocity reference frame, respectively. Input pulses will be generated on the edge of the fibre, i.e. at $t=0$. Thus, to investigate the pulse propagation along the fibre, one should solve the Cauchy problem for the NS equation (4).

According to the inverse scattering technique (IST), to find which type of initial envelope function generates solitons one has to investigate the eigenvalue Zakharov-Shabat problem (Zakharov and Shabat 1972):

$$\begin{aligned}\partial\psi_1/\partial x&=\mathrm{i}\lambda\psi_1+\mathrm{i}u(x,0)\psi_2\\ \partial\psi_2/\partial x&=-\mathrm{i}\lambda\psi_2+\mathrm{i}u^*(x,0)\psi_1\end{aligned} \tag{5}$$

with $u(x,0)$ (which falls off fast enough at $x=\pm\infty$) being the initial condition to (4). Each discrete eigenvalue $\lambda=\xi+\mathrm{i}\eta$ corresponds to a soliton with amplitude 2η moving with velocity -4ξ (see the notations by Zakharov *et al* (1980)). The important information about soliton solutions is contained in the so-called Jost coefficients $a(\lambda)$ and $b(\lambda)$, the scattering amplitudes related to the spectral problem (5). In particular, the zeros of $a(\lambda)$ are the discrete eigenvalues of the spectral problem (5). Therefore, to obtain information about solitons one has to find the Jost coefficient $a(\lambda)$.

Let us consider the special family of the initial potentials

$$u(x,0)=U(x,0)\,\mathrm{e}^{\mathrm{i}\delta} \tag{6}$$

where $U(x,0)$ is a real function and δ $(0\leqslant\delta<2\pi)$ is an arbitrary constant. As was mentioned in the paper by Burzlaff (1988a), if the potential $u(x,0)$ has eigenvalue $\lambda=\xi+\mathrm{i}\eta$, then the Galilei transform

$$\tilde{u}(x,0)\rightarrow u(x,0)\exp(-\mathrm{i}Vx)=U(x,0)\exp(-\mathrm{i}Vx+\mathrm{i}\delta)$$

has eigenvalue $\xi+V/2+\mathrm{i}\eta$. Therefore, solving the eigenvalue problem for $u(x,0)$ yields a solution for the whole family of Galilei transformations as well. So we can concentrate on $u(x,0)$ itself and determine its discrete eigenvalues to formulate the threshold condition for soliton generation. Moreover, it is easy to prove that the transformation

$$\psi_1\rightarrow\Psi_1\,\mathrm{e}^{\mathrm{i}\gamma}\qquad\psi_2\rightarrow\Psi_2\exp[\mathrm{i}(\gamma-\delta)]$$

leads to the most simple eigenvalue problem

$$\begin{aligned}\partial\Psi_1/\partial x&=\mathrm{i}\lambda\Psi_1+\mathrm{i}U(x,0)\Psi_2\\ \partial\Psi_2/\partial x&=-\mathrm{i}\lambda\Psi_2+\mathrm{i}U(x,0)\Psi_1\end{aligned} \tag{5$'$}$$

with a real initial potential $U(x,0)$. The eigenproblem (5′) is the same as in the theory of the exactly integrable sine-Gordon (SG) equation $\partial^2u/\partial t^2-\partial^2u/\partial x^2+\sin u=0$, with the initial conditions $u(x,0)=0$, $\partial u(x,0)/\partial t=4U(x,0)$ and $\lambda\rightarrow\frac{1}{2}(\tilde{\lambda}-1/4\tilde{\lambda})$, $\tilde{\lambda}$ $(0\leqslant\tilde{\lambda}<\infty)$ being the spectral parameter used in the scattering problem related to the SG equation (see details in Zakharov *et al* (1980)).

As was demonstrated in the SG equation theory (Zakharov *et al* 1980) the scattering problem (5′) has the reduction

$$\Psi_1(x, \lambda) = -\Psi_1^*(x, -\lambda^*) \qquad \Psi_2(x, \lambda) = \Psi_2^*(x, -\lambda^*)$$

and

$$a(\lambda) = a^*(-\lambda^*) \qquad b(\lambda) = -b^*(-\lambda^*)$$

where $a(\lambda)$ and $b(\lambda)$ are the Jost coefficients used in the IST. From the relation $a(\lambda) = a^*(-\lambda^*)$ one can conclude that the zeros of $a(\lambda)$ will appear either at $\lambda = 0$ or as pairs $\lambda_{1,2} = \pm\lambda' + i\lambda''$. The former case corresponds to the generation of envelope optical solitons with the group velocity (i.e. with zero velocities in the group-velocity reference frame), and the latter case is related to the generation of soliton pairs with velocities $\pm 4\lambda'$. From a physical viewpoint it is evident that the generation of the single quiescent soliton will occur with a smaller energy than the soliton pair. Therefore, the threshold condition for the soliton generation is determined by properties of $a(\lambda = 0)$ (see also the paper by Shvartsburg and Zuev (1980), where some proofs were obtained†).

The formal solution of the eigenvalue problem (5′) with $\lambda = 0$ has the form

$$\begin{aligned} \Psi_1(x, 0) &= \exp(-iS(x))\left(C_1 \int_{-\infty}^{x} U(x', 0) \exp(2iS(x'))\, dx' + C_2\right) \\ \Psi_2(x, 0) &= -iC_1 \exp(iS(x)) - \Psi_1 \end{aligned} \tag{7}$$

where

$$S(x) = \int_{-\infty}^{x} U(x', 0)\, dx'.$$

Using the solution (7) we may determine the transition matrix (see, e.g., Zakharov *et al* 1980) at $\lambda = 0$, i.e. in particular, we may find the Jost coefficient $a(0)$. If we choose $\Psi_1(x, 0) \to 0$ for $x \to -\infty$, then

$$\begin{aligned} a(0) &= \lim_{x\to+\infty} \Psi_2(x, 0) \\ &= -iC_1\left(\exp(iS_0) - i\exp(-iS_0) \int_{-\infty}^{\infty} U(x, 0) \exp(2iS(x))\, dx\right) \end{aligned}$$

where

$$S_0 \equiv S(x = +\infty) = \int_{-\infty}^{\infty} U(x, 0)\, dx. \tag{8}$$

After straightforward transformations one can obtain (cf the results by Shvartsburg and Zuev (1980))

$$a(0) = -iC_1 \cos S_0. \tag{9}$$

The analysis of the result (9) shows that the threshold condition for the soliton generation may be presented as follows:

$$S_0 \equiv \int_{-\infty}^{\infty} U(x, 0)\, dx \geqslant \pi/2. \tag{10}$$

† It is strongly valid for positively defined input pulses, i.e. for $U(x, 0) > 0$ (for any x) which is usually real in optical fibres.

If the input pulse (6) generates only optical solitons with group velocities then, according to (9), the number of the solitons is

$$N = \langle \tfrac{1}{2} + S_0/\pi \rangle \tag{11}$$

(cf (1) and (2)). For our input pulses (6) with $U(x,0) > 0$ for any x it is valid:

$$S_0 \equiv \int_{-\infty}^{\infty} U(x,0)\,\mathrm{d}x = \int_{-\infty}^{\infty} |u(x,0)|\,\mathrm{d}x \equiv F$$

so that condition (11) is the same as (1). Thus, conditions (10) and (11) may be regarded as general conditions for the soliton generation in optical fibres from the above-mentioned family of input pulses.

Let us consider, for example, the super-Gaussian envelope pulse (3). It is evident that our results may also be applied to this pulse at $\alpha = 0$. Simple calculations yield

$$S_0 = S_0^{(m)} = (2^{1/2m}/m) A_0 \sigma \Gamma(1/2m)$$

$\Gamma(x)$ being the gamma function. Therefore, the condition for soliton generation is $S_0^{(m)} > \pi/2$. For limit values of m one can obtain: $S_0^{(1)} = A_0\sigma(2\pi)^{1/2}$ and $S_0^{(m)} \approx 2A_0\sigma$ for $m \gg 1$.

In conclusion we note that the results obtained above may also be applied to the SG equation with a special type of initial conditions: $u(x,0) = 0$, $\partial u(x,0)/\partial t \neq 0$. In particular, the threshold condition for the soliton generation (in the case of the SG equation such an initial pulse may generate either pairs of SG kinks with opposite polarities or SG breathers (see Zakharov *et al* (1980))) is

$$\int_{-\infty}^{\infty} \frac{\partial u(x,0)}{\partial t}\,\mathrm{d}x \geqslant 2\pi.$$

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